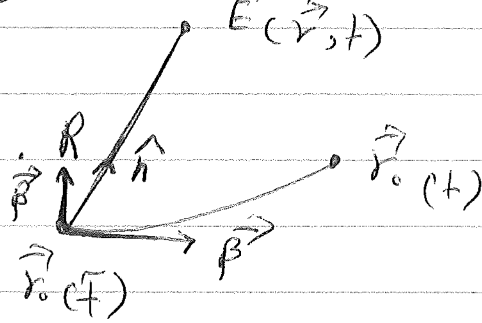
Lec 10: 02/20/2012

Radiative Processes:

Nearly all of our knowledge about astronomical objects is obtained through the radiation they produce. The physical characteristics of their emission region are often inferred from features in their spectra. Since we understand radiation processes rather well, the theoretical modeling of these environments often leads to quantitative results.

Interactions of photons with charged particles is also important for other reasons. Not only these interactions lead to radiation from high-energy charged particles, but also they lead to the energy loss of the latter. This will be important when discussing the spectrom of charged particles reaching us from high-energy sources.

Let us start by considering radiation from a moving charge. In the full relativistic treatment, the electric field of a particle moving at velocity $\vec{v}$ is given by:



$$\vec{E}(\vec{r},t) = \vec{E}_{vel} + \vec{E}_{rad}$$

$$\vec{E}_{vel} = q\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{\tilde{t}} , \qquad \vec{E}_{rad} = \frac{q}{c}\left[\frac{\hat{n}\times((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}})}{\kappa^3 R}\right]_{\tilde{t}}$$

Here:

$$\vec{\beta} = \frac{\vec{v}}{c} , \qquad \kappa \equiv 1-\hat{n}\cdot\vec{\beta} , \qquad \vec{R} = \vec{r}-\vec{r}_0$$

Note that the above expressions are calculated at the retarded time $\tilde{t}$, which is the solution to the following equation:

$$\tilde{t} = t - \frac{|\vec{r}(t)-\vec{r}_0(\tilde{t})|}{c}$$

Also note that $\vec{E}_{vel}$ (called the velocity field) is $\propto \frac{1}{R^2}$ while $\vec{E}_{rad}$ (called the radiation field) is $\propto \frac{1}{R}$. $\vec{E}_{rad} \neq 0$

only when $\dot{\vec{\beta}} \neq 0$, i.e. in the presence of particle acceleration. This happens, for example, when the particle is accelerated in a magnetic field (which is the case in cyclotron radiation) or in the electric field induced by other heavy charged particles (which is the case in Bremsstrahlung).

As always, the magnetic field can be calculated from:

$$\vec{B}_{(\vec{r},t)} = \hat{n} \times \vec{E}_{(\vec{r},t)}$$

Specializing to the non-relativistic limit, $\beta \ll 1$, and considering only $\vec{E}_{rad}$, we have:

$$\vec{E}_{rad} = \frac{q}{Rc^2}\,\hat{n} \times (\hat{n} \times \dot{\vec{v}}) \quad , \quad \vec{B}_{rad} = \hat{n} \times \vec{E}_{rad}$$

Thus,

$$|\vec{E}_{rad}| = |\vec{B}_{rad}| = \frac{q\,\dot{v}}{Rc^2}\sin\theta$$

Here θ is the angle between $\vec{R}$ and $\dot{\vec{v}}$.

The energy flux is given by the Pointing vector:

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} \Rightarrow |\vec{S}| = \frac{c}{4\pi} E_{rad}^2$$

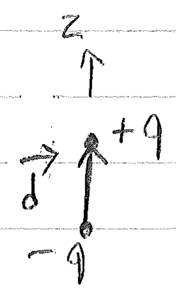
The emitted power within the solid angle $d\Omega$ is:

$$\frac{dP}{d\Omega} = R^2 S = \frac{q^2 \dot{v}^2}{4\pi c^3} \sin^2\theta \Rightarrow P = \frac{2q^2\dot{v}^2}{3c^3}$$

The interesting point is that P depends on $q\dot{v}$, which is the second derivative of the electric dipole moment $\vec{d}$:

$$\vec{d} \equiv q L \hat{z} \Rightarrow \ddot{\vec{d}} = q \dot{\vec{v}} \hat{z}$$

$$P = \frac{2|\ddot{\vec{d}}|^2}{3c^3}$$



This is the familiar dipole approximation for a single non-relativistic particle.

It is also important to know the frequency distribution of the radiation (in addition to its distribution in solid angle).

This can be done by writing $\vec{d}$ and $\vec{E}_{rad}$ in the Fourie space:

$$\vec{d}(t) = \int_{-\infty}^{+\infty} e^{-i\omega t} \vec{\tilde{d}}(\omega)\, d\omega \Rightarrow \ddot{\vec{d}}(t) = -\int_{-\infty}^{+\infty} \omega^2 e^{-i\omega t} \vec{\tilde{d}}(\omega)\, d\omega$$

$$\Rightarrow \vec{E}_{rad}(t) = -\int_{-\infty}^{+\infty} \omega^2 e^{-i\omega t} \vec{\tilde{d}}(\omega) \frac{\sin\theta}{Rc^2}\, d\omega$$

Then:

$$\frac{dP}{dA} = |\vec{S}| = \frac{c}{4\pi} E_{rad}^2(t) \Rightarrow \frac{d^2 W}{dt\, dA} = \frac{c}{4\pi} E_{rad}^2(t) \Rightarrow$$

$$\frac{dW}{dA} = \frac{c}{4\pi}\int_{-\infty}^{+\infty} E_{rad}^2(t)\, dt = \frac{c}{2\pi}\int_{-\infty}^{+\infty} |\vec{\tilde{E}}_{rad}(\omega)|^2\, d\omega$$

Here we have used Parseval's theorem $\int_{-\infty}^{+\infty} E_{rad}^2(t)\, dt = 2\pi\int_{-\infty}^{+\infty} |\vec{\tilde{E}}_{rad}(\omega)|^2\, d\omega$. Also, since $|\vec{\tilde{E}}_{rad}(\omega)|^2 = |\vec{\tilde{E}}_{rad}(-\omega)|^2$, we find:

$$\frac{dW}{dA} = c\int_0^{+\infty} |\vec{\tilde{E}}_{rad}(\omega)|^2\, d\omega$$

In terms of the dipole moment $\vec{\tilde{d}}(\omega)$, it reads:

$$\frac{dW}{dA} = \frac{1}{R^2 c^3}\int_0^{+\infty} \omega^4 |\vec{\tilde{d}}(\omega)|^2 \sin^2\theta\, d\omega \Rightarrow \frac{dW}{d\omega\, d\Omega} = \frac{\omega^4}{c^3} |\vec{\tilde{d}}(\omega)|^2 \sin^2\theta \Rightarrow \frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} |\vec{\tilde{d}}(\omega)|^2$$

$dA = R^2 d\Omega$

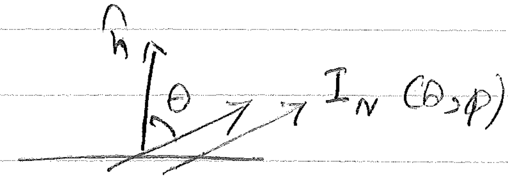
Note the ω^4 dependence of $\frac{dW}{d\omega}$, which is due to $|\vec{S}|$ being dependent on $\dot{v}^2$.

It is also useful to estabish some basic definitions and consider how one transforms a description of the photon field from one frame to another before turning to various radiation mechanisms.

The specific intensity at a frequency ν, denoted by I_ν, is defined by the relation:

$dW = I_\nu \, dA \, dt \, d\Omega \, d\nu$

The intensity represents the energy carried along an individual ray, which may be in any direction relative to the unit vector $\hat{n}$ normal to dA. Thus, the flux density F_ν is given by:

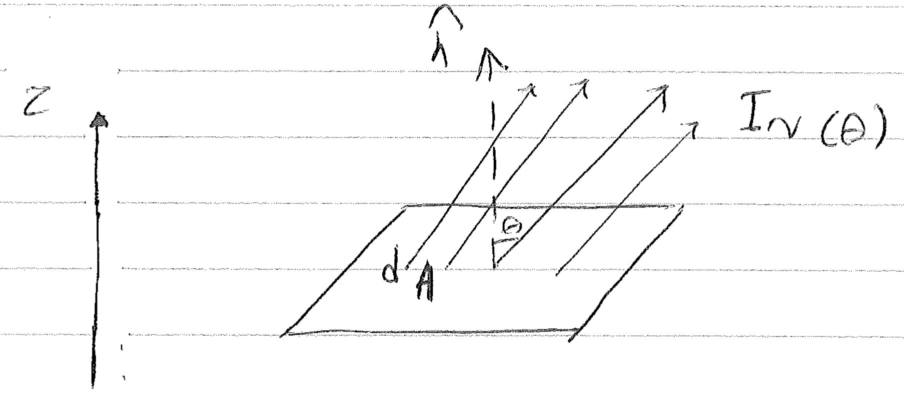


$F_\nu = \int I_\nu(\theta,\phi) \cos\theta \, d\Omega$

The total flux F and intensity I are:

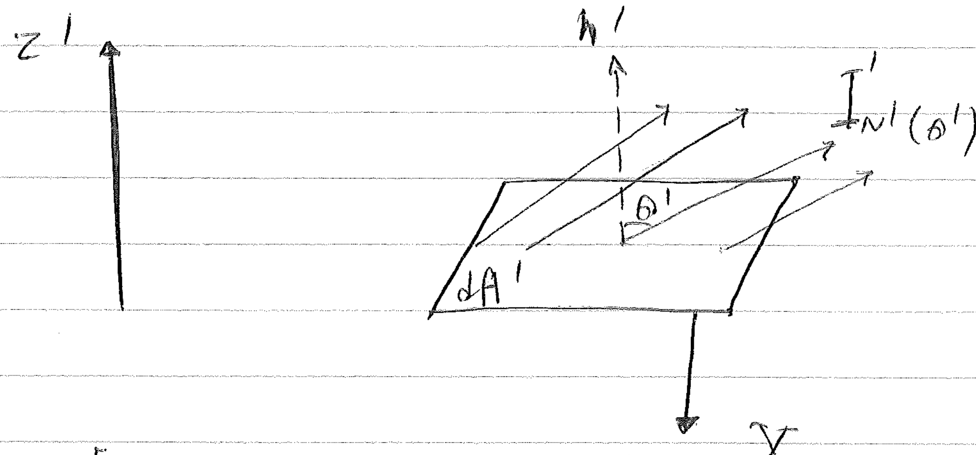
$F = \int F_\nu \, d\nu$, $I = \int I_\nu \, d\nu$

Now let us consider the number of photons N in the frequency interval $(\nu, \nu + d\nu)$ passing through an element of area dA that is perpendicular to $\hat{z}$ and is at rest in laboratory:



$$N = \left[\frac{I_\nu(\theta)}{h\nu}\right] d\Omega \, d\nu \, dA \cos\theta \, dt$$

Note that the energy of an individual photon is $h\nu$. We want to see what this looks like according to a second observer moving at velocity v in the z direction. The element of area moves with velocity v according to this observer, which results in an additional volume $v\, dA'\, dt'$ swept out by the area:



$$N' = \left[\frac{I'_{\nu'}(\theta')}{h\nu'}\right] d\Omega'\, d\nu' \left[dA' \cos\theta'\, dt' + \frac{v}{c}\, dA'\, dt'\right]$$

The total number of photons passing through the element must be the same according to both observers:

$$N = N' \Rightarrow \left[\frac{I_\nu}{\nu}\right] d\Omega\, d\nu\, \cos\theta\, dt = \left[\frac{I'_{\nu'}}{\nu'}\right] d\Omega'\, d\nu' \left[\cos\theta' + \frac{v}{c}\right] dt'$$

Different quantities in the two frames are related through the following expressions:

$$dt' = \gamma \, dt \quad , \quad \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$d\Omega = \sin\theta \, d\theta \, d\phi = \frac{d\Omega'}{\gamma^2 (1 + \beta\cos\theta')^2} \quad , \quad \nu = \gamma(1 + \beta\cos\theta')\,\nu'$$

This results in:

$$\frac{I_\nu}{\nu^3} = \frac{I'_{\nu'}}{\nu'^3}$$

Therefore the quantity $\frac{I_\nu}{\nu^3}$ is a Lorentz-invariant quantity. This quantity finds many uses in high-energy astrophysics, including a direct application to the radiative flux produced in relativistic jets (which will be discussed later).